

Hypersurface family with a common isoasymptotic curve

Ergin Bayram, Emin Kasap
Ondokuz Mayıs University

August 28, 2014

Abstract

In the present paper, we handle the problem of finding a hypersurface family from a given asymptotic curve in \mathbb{R}^4 . Using the Frenet frame of the given asymptotic curve, we express the hypersurface as a linear combination of this frame and analyze the necessary and sufficient conditions for that curve to be asymptotic. We illustrate this method by presenting some examples.

Keywords: Hypersurface, Frenet frame, asymptotic curve.

Mathematics Subject Classification (2010): 53A04, 53A07.

1 Introduction

Asymptotic curves are encountered in differential geometry frequently. A surface curve is called asymptotic if its tangent vectors always point in an asymptotic direction, that is, the direction in which the normal curvature is zero. In an asymptotic direction, the surface is not bending away from its tangent plane.

Asymptotic curves on a surface can be seen in many differential geometry books [17] – [21]. Rastogi [18] obtained the differential equation of hyperasymptotic curves by a new method and showed some properties of these curves. Aminov [3] established more general expressions for the curvature of asymptotic curves of submanifolds in the Riemannian space. Romero - Fuster et.al. [19] studied asymptotic curves on generally immersed surfaces in \mathbb{R}^5 . Both the general and rational developable surface pencils through an arbitrary parametric curve as its common asymptotic curve were analyzed by Liu and Wang [16]. Bayram et. al. [6] tackled the problem of finding a surface pencil from a given asymptotic curve.

However, while differential geometry of a parametric surface in \mathbb{R}^3 can be found in textbooks such as in Struik [21], Willmore [24], Stoker [20], do Carmo [7], differential geometry of a parametric surface in \mathbb{R}^n can be found in textbooks such as in the contemporary literature on Geometric Modeling [9], [13]. Also, there is little literature on differential geometry of parametric surface family in

\mathbb{R}^3 [2], [8], [14], [22], [6], but not in \mathbb{R}^4 . Besides, there is an ascending interest on fourth dimension [1], [2], [8].

Furthermore, various visualization techniques about objects in Euclidean n -space ($n \geq 4$) are presented [5], [4], [11]. The fundamental step to visualize a 4D object is projecting first into the 3-space and then into the plane. In many real world applications, the problem of visualizing three-dimensional data, commonly referred to as scalar fields arouses. The graph of a function $\mathbf{f}(x, y, z) : U \subset \mathbb{R}^3 \rightarrow \mathbb{R}$, where U is open, is a special type of parametric hypersurface with the parametrization $(x, y, z, \mathbf{f}(x, y, z))$ in 4-space. There exists a method for rendering such a 3-surface based on known methods for visualizing functions of two variables [10].

In this paper, we consider the four dimensional analogue problem of constructing a parametric representation of a surface family from a given asymptotic as in Bayram et al. [6], who derived the necessary and sufficient conditions on the marching-scale functions for which the curve C is an asymptotic curve on a given surface. We express the hypersurface pencil parametrically with the help of the Frenet frame $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ of the given curve. We find the necessary and sufficient constraints on the marching-scale functions, namely, coefficients of Frenet vectors, so that both the asymptotic and parametric requirements are met.

2 Preliminaries

Let us first introduce some notations and definitions. Bold letters such as \mathbf{a} , \mathbf{R} will be used for vectors and vector functions. We assume that they are smooth enough so that all the (partial) derivatives given in the paper are meaningful. Let $\boldsymbol{\alpha} : \mathbf{I} \subset \mathbb{R} \rightarrow \mathbb{R}^4$ be an arc-length curve. If $\{\mathbf{T}, \mathbf{N}, \mathbf{B}_1, \mathbf{B}_2\}$ is the moving Frenet frame along $\boldsymbol{\alpha}$, then the Frenet formulas are given by

$$\begin{cases} \mathbf{T}' = \kappa_1 \mathbf{N}, \\ \mathbf{N}' = -\kappa_1 \mathbf{T} + \kappa_2 \mathbf{B}_1, \\ \mathbf{B}_1' = -\kappa_2 \mathbf{N} + \kappa_3 \mathbf{B}_2, \\ \mathbf{B}_2' = -\kappa_3 \mathbf{B}_1, \end{cases} \quad (1)$$

where $\mathbf{T}, \mathbf{N}, \mathbf{B}_1$ and \mathbf{B}_2 denote the tangent, principal normal, first binormal and second binormal vector fields, respectively, κ_i ($i = 1, 2, 3$) the i -th curvature functions of the curve $\boldsymbol{\alpha}$ [11].

From elementary differential geometry we have

$$\begin{cases} \boldsymbol{\alpha}'(s) = \mathbf{T}(s), \\ \boldsymbol{\alpha}''(s) = \kappa_1(s) \mathbf{N}(s), \\ \kappa_1(s) = \|\boldsymbol{\alpha}''(s)\|. \end{cases} \quad (2)$$

Using Frenet formulas one can obtain the followings

$$\begin{cases} \boldsymbol{\alpha}'''(s) = -\kappa_1^2 \mathbf{T}(s) + \kappa_1' \mathbf{N}(s) + \kappa_1 \kappa_2 \mathbf{B}_1(s), \\ \boldsymbol{\alpha}^{(iv)}(s) = -3\kappa_1 \kappa_1' \mathbf{T}(s) + (-\kappa_1^3 + \kappa_1'' - \kappa_1 \kappa_2^2) \mathbf{N}(s) \\ \quad + (2\kappa_1' \kappa_2 + \kappa_1 \kappa_2') \mathbf{B}_1(s) + \kappa_1 \kappa_2 \kappa_3 \mathbf{B}_2(s). \end{cases} \quad (3)$$

The unit vectors \mathbf{B}_2 and \mathbf{B}_1 are given by

$$\begin{cases} \mathbf{B}_2(s) = \frac{\boldsymbol{\alpha}'(s) \otimes \boldsymbol{\alpha}''(s) \otimes \boldsymbol{\alpha}'''(s)}{\|\boldsymbol{\alpha}'(s) \otimes \boldsymbol{\alpha}''(s) \otimes \boldsymbol{\alpha}'''(s)\|}, \\ \mathbf{B}_1(s) = \mathbf{B}_2(s) \otimes \mathbf{T}(s) \otimes \mathbf{N}(s), \end{cases} \quad (4)$$

where \otimes is the vector product of vectors in \mathbb{R}^4 .

Since the vectors \mathbf{T} , \mathbf{N} , \mathbf{B}_1 , \mathbf{B}_2 are orthonormal, the second curvature κ_2 and the third curvature κ_3 can be obtained from (3) as

$$\begin{cases} \kappa_2(s) = \frac{\mathbf{B}_1(s) \bullet \boldsymbol{\alpha}'''(s)}{\kappa_1(s)}, \\ \kappa_3(s) = \frac{\mathbf{B}_2(s) \bullet \boldsymbol{\alpha}^{(iv)}(s)}{\kappa_1(s) \kappa_2(s)}, \end{cases} \quad (5)$$

where ' \bullet ' denotes the standard inner product.

Let $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_4\}$ be the standard basis for four-dimensional Euclidean space \mathbb{R}^4 . The vector product of the vectors $\mathbf{u} = \sum_{i=1}^4 u_i \mathbf{e}_i$, $\mathbf{v} = \sum_{i=1}^4 v_i \mathbf{e}_i$, $\mathbf{w} = \sum_{i=1}^4 w_i \mathbf{e}_i$ is defined by

$$\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w} = \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 & \mathbf{e}_4 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \\ w_1 & w_2 & w_3 & w_4 \end{vmatrix}$$

[12], [23].

If \mathbf{u} , \mathbf{v} and \mathbf{w} are linearly independent then $\mathbf{u} \otimes \mathbf{v} \otimes \mathbf{w}$ is orthogonal to each of these vectors.

3 Hypersurface Family with a Common Isoasymptotic

A curve $\mathbf{r}(s)$ on a hypersurface $\mathbf{P} = \mathbf{P}(s, t, q) \subset \mathbb{R}^4$ is called an isoparametric curve if it is a parameter curve, that is, there exists a pair of parameters t_0 and q_0 such that $\mathbf{r}(s) = \mathbf{P}(s, t_0, q_0)$. Given a parametric curve $\mathbf{r}(s)$, it is called an *isoasymptotic* of a hypersurface \mathbf{P} if it is both a asymptotic and an isoparametric curve on \mathbf{P} .

Let $C : \mathbf{r} = \mathbf{r}(s)$, $L_1 \leq s \leq L_2$, be a C^3 curve, where s is the arc-length. To have a well-defined principal normal, assume that $\mathbf{r}''(s) \neq 0$, $L_1 \leq s \leq L_2$.

Let $\mathbf{T}(s)$, $\mathbf{N}(s)$, $\mathbf{B}_1(s)$, $\mathbf{B}_2(s)$ be the tangent, principal normal, first binormal, second binormal, respectively; and let $\kappa_1(s)$, $\kappa_2(s)$ and $\kappa_3(s)$ be the first, the second and the third curvature, respectively. Since $\{\mathbf{T}(s), \mathbf{N}(s), \mathbf{B}_1(s), \mathbf{B}_2(s)\}$ is an orthogonal coordinate frame on $\mathbf{r}(s)$ the parametric hypersurface $\mathbf{P}(s, t, q) : [L_1, L_2] \times [T_1, T_2] \times [Q_1, Q_2] \rightarrow \mathbb{R}^4$ passing through $\mathbf{r}(s)$ can be defined as follows:

$$\mathbf{P}(s, t, q) = \mathbf{r}(s) + (\mathbf{u}(s, t, q), \mathbf{v}(s, t, q), \mathbf{w}(s, t, q), \mathbf{x}(s, t, q)) \begin{pmatrix} \mathbf{T}(s) \\ \mathbf{N}(s) \\ \mathbf{B}_1(s) \\ \mathbf{B}_2(s) \end{pmatrix}, \quad (6)$$

$$L_1 \leq s \leq L_2, \quad T_1 \leq s \leq T_2, \quad Q_1 \leq s \leq Q_2,$$

where $\mathbf{u}(s, t, q)$, $\mathbf{v}(s, t, q)$, $\mathbf{w}(s, t, q)$ and $\mathbf{x}(s, t, q)$ are all C^4 functions. These functions are called the *marching scale functions*.

We try to find out the necessary and sufficient conditions for which a hypersurface $\mathbf{P} = \mathbf{P}(s, t, q)$ has the curve C as an isoasymptotic.

First, to satisfy the isoparametricity condition there should exist $t_0 \in [T_1, T_2]$ and $q_0 \in [Q_1, Q_2]$ such that $\mathbf{P}(s, t_0, q_0) = \mathbf{r}(s)$, $L_1 \leq s \leq L_2$, that is,

$$\begin{cases} \mathbf{u}(s, t_0, q_0) = \mathbf{v}(s, t_0, q_0) = \mathbf{w}(s, t_0, q_0) = \mathbf{x}(s, t_0, q_0) \equiv 0, \\ t_0 \in [T_1, T_2], \quad q_0 \in [Q_1, Q_2], \quad L_1 \leq s \leq L_2. \end{cases} \quad (7)$$

Secondly, the curve C is an asymptotic curve on the hypersurface $\mathbf{P}(s, t, q)$ if and only if the normal curvature $\kappa_n = S(T) \bullet T = 0$ along the curve, where S is the shape operator and T is the tangent vector to the curve. The normal $\hat{\mathbf{n}}(s, t_0, q_0)$ of the hypersurface can be obtained by calculating the vector product of the partial derivatives and using the Frenet formula as follows

$$\begin{aligned} \frac{\partial \mathbf{P}(s, t, q)}{\partial s} &= \left(1 + \frac{\partial \mathbf{u}(s, t, q)}{\partial s} - \mathbf{v}(s, t, q) \kappa_1(s)\right) \mathbf{T}(s) \\ &+ \left(\mathbf{u}(s, t, q) \kappa_1(s) + \frac{\partial \mathbf{v}(s, t, q)}{\partial s} - \mathbf{w}(s, t, q) \kappa_2(s)\right) \mathbf{N}(s) \\ &+ \left(\mathbf{v}(s, t, q) \kappa_2(s) + \frac{\partial \mathbf{w}(s, t, q)}{\partial s} - \mathbf{x}(s, t, q) \kappa_3(s)\right) \mathbf{B}_1(s) \\ &+ \left(\mathbf{w}(s, t, q) \kappa_3(s) + \frac{\partial \mathbf{x}(s, t, q)}{\partial s}\right) \mathbf{B}_2(s), \\ \frac{\partial \mathbf{P}(s, t, q)}{\partial t} &= \frac{\partial \mathbf{u}(s, t, q)}{\partial t} \mathbf{T}(s) + \frac{\partial \mathbf{v}(s, t, q)}{\partial t} \mathbf{N}(s) + \frac{\partial \mathbf{w}(s, t, q)}{\partial t} \mathbf{B}_1(s) + \frac{\partial \mathbf{x}(s, t, q)}{\partial t} \mathbf{B}_2(s), \end{aligned}$$

and

$$\frac{\partial \mathbf{P}(s, t, q)}{\partial q} = \frac{\partial \mathbf{u}(s, t, q)}{\partial q} \mathbf{T}(s) + \frac{\partial \mathbf{v}(s, t, q)}{\partial q} \mathbf{N}(s) + \frac{\partial \mathbf{w}(s, t, q)}{\partial q} \mathbf{B}_1(s) + \frac{\partial \mathbf{x}(s, t, q)}{\partial q} \mathbf{B}_2(s).$$

Remark 1 Because,

$$\begin{cases} \mathbf{u}(s, t_0, q_0) = \mathbf{v}(s, t_0, q_0) = \mathbf{w}(s, t_0, q_0) = \mathbf{x}(s, t_0, q_0) \equiv 0, \\ t_0 \in [T_1, T_2], \quad q_0 \in [Q_1, Q_2], \quad L_1 \leq s \leq L_2. \end{cases}$$

along the curve C , by the definition of partial differentiation we have

$$\begin{cases} \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial s} = \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial s} = \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial s} = \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial s} \equiv 0, \\ t_0 \in [T_1, T_2], \quad q_0 \in [Q_1, Q_2], \quad L_1 \leq s \leq L_2. \end{cases}$$

Using (7) we have

$$\begin{aligned} \hat{\mathbf{n}}(s, t_0, q_0) &= \frac{\partial \mathbf{P}(s, t_0, q_0)}{\partial s} \otimes \frac{\partial \mathbf{P}(s, t_0, q_0)}{\partial t} \otimes \frac{\partial \mathbf{P}(s, t_0, q_0)}{\partial q} \\ &= \phi_1(s, t_0, q_0) \mathbf{T}(s) - \phi_2(s, t_0, q_0) \mathbf{N}(s) \\ &\quad + \phi_3(s, t_0, q_0) \mathbf{B}_1(s) - \phi_4(s, t_0, q_0) \mathbf{B}_2(s), \end{aligned}$$

where

$$\begin{aligned}
\phi_1(s, t_0, q_0) &= \left| \begin{array}{ccc} \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial s} \\ \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial t} \\ \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial q} \end{array} \right| = 0, \\
\phi_2(s, t_0, q_0) &= \left| \begin{array}{ccc} 1 + \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial s} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial q} \end{array} \right| \\
&= \left| \begin{array}{ccc} 1 & 0 & 0 \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial q} \end{array} \right| \\
&= \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial t} \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial q} - \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial q} \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial t}, \\
\phi_3(s, t_0, q_0) &= \left| \begin{array}{ccc} 1 + \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial s} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial q} \end{array} \right| \\
&= \left| \begin{array}{ccc} 1 & 0 & 0 \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial q} \end{array} \right| \\
&= \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial t} \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial q} - \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial q} \frac{\partial \mathbf{x}(s, t_0, q_0)}{\partial t}, \\
\phi_4(s, t_0, q_0) &= \left| \begin{array}{ccc} 1 + \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial s} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial s} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial q} \end{array} \right| \\
&= \left| \begin{array}{ccc} 1 & 0 & 0 \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial t} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial t} \\ \frac{\partial \mathbf{u}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial q} & \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial q} \end{array} \right| \\
&= \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial t} \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial q} - \frac{\partial \mathbf{v}(s, t_0, q_0)}{\partial q} \frac{\partial \mathbf{w}(s, t_0, q_0)}{\partial t}.
\end{aligned}$$

So,

$$\begin{aligned}
\kappa_n &= S(T) \bullet T = 0 \Leftrightarrow \hat{\mathbf{n}} \bullet \mathbf{N} = \mathbf{0} \Leftrightarrow \\
\phi_2(s, t_0, q_0) &\equiv 0, \quad \phi_3^2(s, t_0, q_0) + \phi_4^2(s, t_0, q_0) \neq 0, \\
t_0 &\in [T_1, T_2], \quad q_0 \in [Q_1, Q_2], \quad L_1 \leq s \leq L_2.
\end{aligned} \tag{8}$$

Thus, any hypersurface defined by (6) has the curve C as an isoasymptotic if and only if

$$\begin{cases} \mathbf{u}(s, t_0, q_0) = \mathbf{v}(s, t_0, q_0) = \mathbf{w}(s, t_0, q_0) = \mathbf{x}(s, t_0, q_0) \equiv 0, \\ \phi_2(s, t_0, q_0) \equiv 0, \quad \phi_3^2(s, t_0, q_0) + \phi_4^2(s, t_0, q_0) \neq 0, \end{cases} \tag{9}$$

$$t_0 \in [T_1, T_2], \quad q_0 \in [Q_1, Q_2], \quad L_1 \leq s \leq L_2.$$

is satisfied. We call the set of hypersurfaces defined by (6) and satisfying (9) an *isoasymptotic hypersurface family*.

4 Examples

Example 2 Let $\mathbf{r}(s) = \left(\frac{1}{2} \cos(s), \frac{1}{2} \sin(s), \frac{1}{2}s, \frac{\sqrt{2}}{2}s\right)$, $0 \leq s \leq 2\pi$, be a curve parametrized by arc-length. For this curve,

$$\begin{aligned} \mathbf{T}(s) &= \mathbf{r}'(s) = \left(-\frac{1}{2} \sin(s), \frac{1}{2} \cos(s), \frac{1}{2}, \frac{\sqrt{2}}{2}\right), \\ \mathbf{N}(s) &= (-\cos(s), -\sin(s), 0, 0), \\ \mathbf{B}_2(s) &= \frac{\mathbf{r}'(s) \otimes \mathbf{r}''(s) \otimes \mathbf{r}'''(s)}{\|\mathbf{r}'(s) \otimes \mathbf{r}''(s) \otimes \mathbf{r}'''(s)\|} = \left(0, 0, \frac{\sqrt{6}}{3}, -\frac{\sqrt{3}}{3}\right), \\ \mathbf{B}_1(s) &= \mathbf{B}_2(s) \otimes \mathbf{T}(s) \otimes \mathbf{N}(s) = \left(-\frac{\sqrt{3}}{2} \sin(s), \frac{\sqrt{3}}{2} \cos(s), -\frac{\sqrt{3}}{6}, -\frac{\sqrt{6}}{6}\right). \end{aligned}$$

Let us choose the marching-scale functions as

$$\begin{aligned} \mathbf{u}(s, t, q) &= (t - t_0)(q - q_0), \quad \mathbf{v}(s, t, q) = t - t_0, \quad \mathbf{w}(s, t, q) \equiv 0, \quad \mathbf{x}(s, t, q) = q - q_0, \\ t_0 &\in [0, 1], \quad q_0 \in [0, 1], \quad 0 \leq s \leq 2\pi. \end{aligned}$$

So, we have the hypersurface

$$\begin{aligned} \mathbf{P}(s, t, q) &= \mathbf{r}(s) + \mathbf{u}(s, t, q) \mathbf{T}(s) + \mathbf{v}(s, t, q) \mathbf{N}(s) + \mathbf{w}(s, t, q) \mathbf{B}_1(s) + \mathbf{x}(s, t, q) \mathbf{B}_2(s) \\ &= \left(\frac{1}{2} \cos(s) - \frac{1}{2} (t - t_0)(q - q_0) \sin(s) - (t - t_0) \cos(s), \right. \\ &\quad \frac{1}{2} \sin(s) + \frac{1}{2} (t - t_0)(q - q_0) \cos(s) - (t - t_0) \sin(s), \\ &\quad \frac{1}{2}s + \frac{1}{2} (t - t_0)(q - q_0) + \frac{\sqrt{6}}{3} (q - q_0), \\ &\quad \left. \frac{\sqrt{2}}{2}s + \frac{\sqrt{2}}{2} (t - t_0)(q - q_0) - \frac{\sqrt{3}}{3} (q - q_0) \right), \end{aligned}$$

$0 \leq s \leq 2\pi$, $0 \leq t \leq 1$, $0 \leq q \leq 1$, $t_0 \in [0, 1]$, $q_0 \in [0, 1]$, is a member of the *isoasymptotic hypersurface family*, since it satisfies (9).

By changing the parameters t_0 and q_0 we can adjust the position of the curve $\mathbf{r}(s)$ on the hypersurface. Let us choose $t_0 = \frac{1}{2}$ and $q_0 = 0$. Now the curve $\mathbf{r}(s)$

is again an isoasymptotic on the hypersurface $\mathbf{P}(s, t, q)$ and the equation of the hypersurface is

$$\begin{aligned} \mathbf{P}(s, t, q) = & \left(\cos s - t \cos s + \frac{1}{4}q \sin s - \frac{1}{2}qt \sin s, \right. \\ & \sin s - \frac{1}{4}q \cos s - t \sin s + \frac{1}{2}qt \cos s, \\ & \frac{1}{2}s - \frac{1}{4}q + \frac{1}{3}\sqrt{6}q + \frac{1}{2}qt, \\ & \left. \frac{1}{2}\sqrt{2}s - \frac{1}{3}\sqrt{3}q - \frac{1}{4}\sqrt{2}q + \frac{1}{2}\sqrt{2}qt \right). \end{aligned}$$

The projection of a hypersurface into 3-space generally yields a three-dimensional volume. If we fix each of the three parameters, one at a time, we obtain three distinct families of 2-spaces in 4-space. The projections of these 2-surfaces into 3-space are surfaces in 3-space. Thus, they can be displayed by 3D rendering methods.

So, if we (parallel) project the hypersurface $\mathbf{P}(s, t, q)$ into the $\mathbf{w} = 0$ subspace and fix $q = 0$ we obtain the surface

$$\mathbf{P}_{\mathbf{w}}(s, t, 0) = \left(\cos s - t \cos s, \sin s - t \sin s, \frac{1}{2}s \right),$$

$0 \leq s \leq 2\pi, 0 \leq t \leq 1$ in 3-space illustrated in Fig. 1.

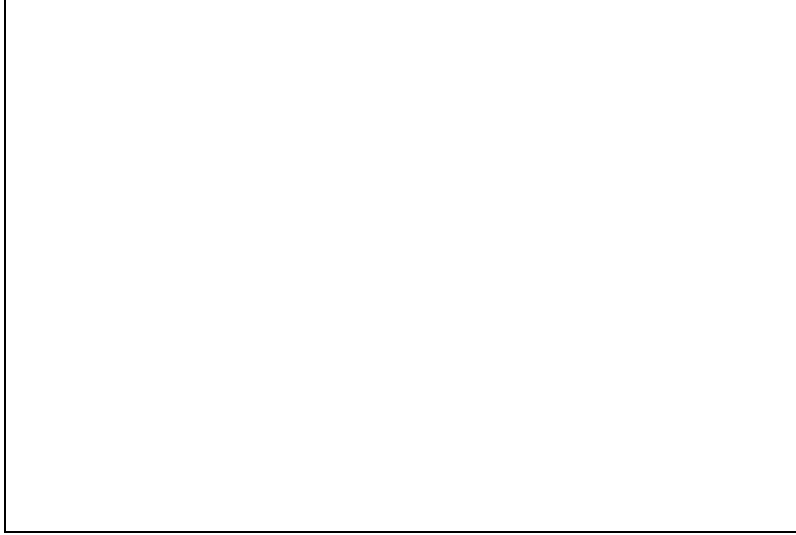


Fig. 1. Projection of a member of the hypersurface family and its isoasymptotic.

Example 3 Given the curve $\mathbf{r}(s) = \left(\frac{1}{2} \sin(s), \frac{1}{2} \cos(s), 0, \frac{\sqrt{3}}{2}s \right)$, $0 \leq s \leq 3$,

it is easy to show that

$$\begin{aligned}
\mathbf{T}(s) &= \mathbf{r}'(s) = \left(\frac{1}{2} \cos(s), -\frac{1}{2} \sin(s), 0, \frac{\sqrt{3}}{2} \right), \\
\mathbf{N}(s) &= (-\sin(s), -\cos(s), 0, 0), \\
\mathbf{B}_2(s) &= \frac{\mathbf{r}'(s) \otimes \mathbf{r}''(s) \otimes \mathbf{r}'''(s)}{\|\mathbf{r}'(s) \otimes \mathbf{r}''(s) \otimes \mathbf{r}'''(s)\|} = (0, 0, -1, 0), \\
\mathbf{B}_1(s) &= \mathbf{B}_2 \otimes \mathbf{T} \otimes \mathbf{N} = \left(\frac{\sqrt{3}}{2} \cos(s), -\frac{\sqrt{3}}{2} \sin(s), 0, -\frac{1}{2} \right).
\end{aligned}$$

Let us choose the marching-scale functions as

$$\begin{aligned}
\mathbf{u}(s, t, q) &= (t - t_0), \\
\mathbf{v}(s, t, q) &= (s + t + 1)(q - q_0), \\
\mathbf{w}(s, t, q) &\equiv 0, \\
\mathbf{x}(s, t, q) &= (s + 1)(t - t_0).
\end{aligned}$$

From (9), the hypersurface

$$\begin{aligned}
\mathbf{P}(s, t, q) &= \mathbf{r}(s) + \mathbf{u}(s, t, q) \mathbf{T}(s) + \mathbf{v}(s, t, q) \mathbf{N}(s) + \mathbf{w}(s, t, q) \mathbf{B}_1(s) + \mathbf{x}(s, t, q) \mathbf{B}_2(s) \\
&= \left(\frac{1}{2} \sin(s) - (s + t + 1)(q - q_0) \sin(s) + \frac{1}{2}(t - t_0) \cos(s), \right. \\
&\quad \frac{1}{2} \cos(s) - (s + t + 1)(q - q_0) \cos(s) - \frac{1}{2}(t - t_0) \sin(s), \\
&\quad \left. - (s + 1)(t - t_0), \right. \\
&\quad \left. \frac{\sqrt{3}}{2}s + \frac{\sqrt{3}}{2}(t - t_0) \right),
\end{aligned}$$

$0 \leq s \leq 3$, $0 \leq t \leq 1$, $0 \leq q \leq 1$, is a member of the hypersurface family having the curve $\mathbf{r}(s)$ as an isoasymptotic.

Setting $t_0 = \frac{1}{2}$ and $q_0 = 0$ yields the hypersurface

$$\begin{aligned}
\mathbf{P}(s, t, q) &= \left(\frac{1}{2} \sin(s) - (s + t + 1)q \sin(s) + \frac{1}{2} \left(t - \frac{1}{2} \right) \cos(s), \right. \\
&\quad \frac{1}{2} \cos(s) - (s + t + 1)q \cos(s) - \frac{1}{2} \left(t - \frac{1}{2} \right) \sin(s), \\
&\quad \left. - (s + 1) \left(t - \frac{1}{2} \right), \frac{\sqrt{3}}{2}s + \frac{\sqrt{3}}{2} \left(t - \frac{1}{2} \right) \right),
\end{aligned}$$

By (parallel) projecting the hypersurface $\mathbf{P}(s, t, q)$ into the subspace $\mathbf{w} = \mathbf{0}$ and

fixing $q = 0$ we get the surface

$$\begin{aligned}\mathbf{P}_{\mathbf{w}}(s, t, 0) = & \left(\frac{1}{2} \sin(s) + \frac{1}{2} \left(t - \frac{1}{2} \right) \cos(s), \right. \\ & \frac{1}{2} \cos(s) - \frac{1}{2} \left(t - \frac{1}{2} \right) \sin(s), \\ & \left. - (s + 1) \left(t - \frac{1}{2} \right) \right),\end{aligned}$$

where, $0 \leq s \leq 3$, $0 \leq t \leq 1$ in 3-space, illustrated in Fig. 2.



Fig. 2. Projection of a member of the hypersurface family and its isoasymptotic.

For the same curve in question let us choose marching-scale functions as

$$\begin{aligned}\mathbf{u}(s, t, q) & \equiv 0, \\ \mathbf{v}(s, t, q) & = \sin(s(q - q_0)), \\ \mathbf{w}(s, t, q) & \equiv 0, \\ \mathbf{x}(s, t, q) & = sq^2(t - t_0).\end{aligned}$$

Thus, from (9) the curve $\mathbf{r}(s)$ is an isoasymptotic on the hypersurface

$$\begin{aligned}\mathbf{P}(s, t, q) & = \mathbf{r}(s) + \mathbf{u}(s, t, q) \mathbf{T}(s) + \mathbf{v}(s, t, q) \mathbf{N}(s) + \mathbf{w}(s, t, q) \mathbf{B}_1(s) + \mathbf{x}(s, t, q) \mathbf{B}_2(s) \\ & = \left(\frac{1}{2} \sin(s) - \sin(s) \sin(s(q - q_0)), \right. \\ & \quad \frac{1}{2} \cos(s) - \cos(s) \sin(s(q - q_0)), \\ & \quad \left. -sq^2(t - t_0), \frac{\sqrt{3}}{2}s \right),\end{aligned}$$

where $0 < s \leq \frac{\pi}{2}$, $0 \leq t \leq 1$, $0 < q < 1$.

By taking $t_0 = 1$ and $q_0 = \frac{1}{2}$ we have the following hypersurface:

$$\begin{aligned} \mathbf{P}(s, t, q) = & \left(\frac{1}{2} \sin(s) - \sin(s) \sin\left(s \left(q - \frac{1}{2}\right)\right), \right. \\ & \frac{1}{2} \cos(s) - \cos(s) \sin\left(s \left(q - \frac{1}{2}\right)\right), \\ & \left. -sq^2(t-1), \frac{\sqrt{3}}{2}s \right). \end{aligned}$$

Hence, if we (parallel) project the hypersurface $\mathbf{P}(s, t, q)$ into the $\mathbf{z} = \mathbf{0}$ subspace we get the surface

$$\begin{aligned} \mathbf{P}_z(s, q) = & \left(\frac{1}{2} \sin(s) - \sin(s) \sin\left(s \left(q - \frac{1}{2}\right)\right), \right. \\ & \frac{1}{2} \cos(s) - \cos(s) \sin\left(s \left(q - \frac{1}{2}\right)\right), \\ & \left. \frac{\sqrt{3}}{2}s \right), \end{aligned}$$

where $0 < s \leq \frac{\pi}{2}$, $0 < q < 1$, in 3-space shown in Fig. 3.

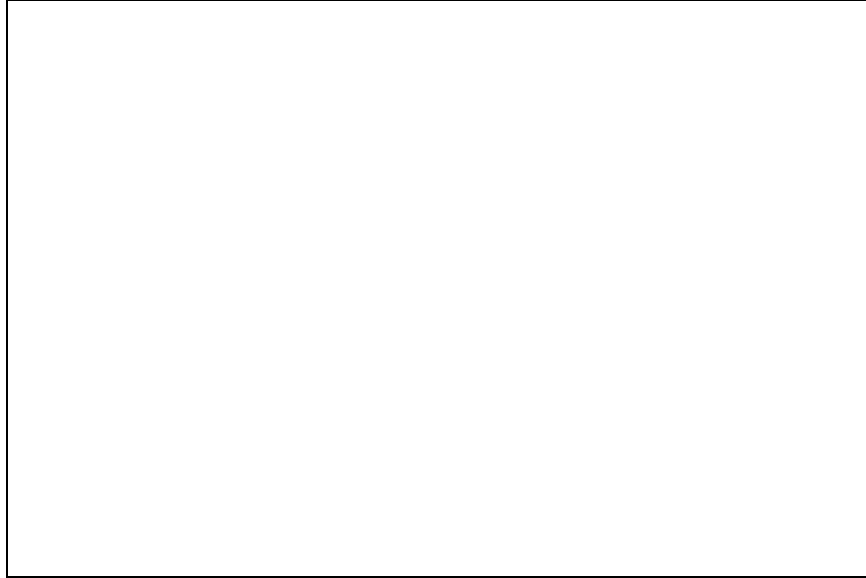


Fig. 3. Projection of a member of the hypersurface family and its isoasymptotic.

References

- [1] Abdel-All N. H., Badr S. A., Soliman M. A., Hassan S.A., Intersection curves of hypersurfaces in \mathbb{R}^4 , Comput. Aided Geom. Des., 29, 99 - 108, 2012.
- [2] Aléssio O., Differential geometry of intersection curves in \mathbb{R}^4 of three implicit surfaces, Comput. Aided Geom. Des.26, 455 – 471, 2009.
- [3] Aminov Yu. A., Asymptotic curves of submanifolds, Mathematical Notes, Vol. 52 (5), 1081 - 1087, 1992.
- [4] Banchoff T. F., Discovering the fourth dimension, Prime Computer, Inc., Natick, MA, 1987.
- [5] Banchoff T. F., Beyond the third dimension: geometry, computer graphics, and higher dimensions, W.H. Freeman & Co., New York, NY, USA,1990.
- [6] Bayram E., Güler F., Kasap E., Parametric representation of a surface pencil with a common asymptotic curve, Computer Aided Design, Vol. 44 (7), 637 - 643, 2012.
- [7] do Carmo M. P., Differential geometry of curves and surfaces, Prentice Hall, Englewood Cliffs, NJ, 1976.
- [8] Düldül M., On the intersection curve of three parametric hypersurfaces, Comput. Aided Geom. Des.27, 118 - 127, 2010.
- [9] Farin G., Curves and surfaces for computer aided geometric design: a practical guide, Academic Press, Inc., San Diego, CA, 2002.
- [10] Hamann B., Visualization and modeling contours of trivariate functions, Ph.D. thesis, Arizona State University, 1991.
- [11] Hanson A. J., Heng P. A., Visualizing the fourth dimension using geometry and light, In: Proceedings of the 2nd Conference on Visualization '91, IEEE Computer Society Press, Los Alamitos, CA, USA, pp. 321–328, 1991.
- [12] Hollasch S. R., Four-space visualization of 4D objects, Master thesis, Arizona State University, 1991.
- [13] Hoschek J., Lasser D., Fundamentals of computer aided geometric design, A.K. Peters, Wellesley, MA, 1993.
- [14] Kasap E., Akyıldız F. T., Orbay K., A generalization of surfaces family with common spatial geodesic, Appl. Math. Comput., 201,781 - 789, 2008.
- [15] Klingenberg W., A course in differential geometry, New York: Springer Verlag: 1978.

- [16] Liu Y., Wang G. J., Designing developable surface pencil through given curve as its common asymptotic curve, Journal of Zhejiang University (Engineering Science), Vol. 47 (7), 1246 - 1252, 2013.
- [17] O'Neill B., Elementary differential geometry, New York: Academic Press Inc.: 1966.
- [18] Rastogi S. C., Hyper-asymptotic curves of a Riemannian hypersurface, İstanbul University Science Faculty Journal of Mathematics, Physics and Astronomy, Vol. 34, 15 - 19, 1969.
- [19] Romero - Fuster M. C., Ruas M. A. S., Tari F., Asymptotic curves in \mathbb{R}^5 , Communications in Contemporary Mathematics, Vol. 10, No.3, 309 - 335, 2008.
- [20] Stoker J. J., Differential geometry, Wiley, New York, 1969.
- [21] Struik D. J., Lectures on classical differential geometry, New York: Dover Publications Inc: 1961.
- [22] Wang G. J., Tang K., Tai C.L., Parametric representation of a surface pencil with a common spatial geodesic, Comput. Aided Des., 36, 447 - 459, 2004.
- [23] Williams M. Z., Stein F. M., A triple product of vectors in four-space, Math. Mag., 37, 230 - 235, 1964.
- [24] Willmore T. J., An introduction to differential geometry, Clarendon Press, Oxford, 1959.